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One-Sided Approximation of Functions

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1. INTRODUCTION

It frequently occurs that a problem of approximation in a normed linear space has associated with it a dual problem of maximizing functionals. The simplest example is when Y is a subspace of a normed linear space X, then for $x_0 \in X$

$$\inf\{\|x_0 - y\| : y \in Y\} = \sup\{L(x_0) : L \in Y^{\perp}, \|L\| = 1\}$$

where $Y^{\perp}$ denotes the orthogonal complement of Y (see [*1*]).

Another example of this type of duality appears in one of the classical moment problems. Let f be Borel measurable and bounded on $[a,b]$, $h_1, \ldots, h_n$ a Čebyšev system on $[a,b]$ and for a given positive Borel measure γ_0 defined on $[a,b]$ denote by $\sum(\gamma_0)$ the class of all positive Borel measures γ which satisfy

$$\int_a^b h_k\, d\gamma = \int_b^a h_k\, d\gamma_0 \quad k = 1, 2, \ldots, n.$$

We then have

$$\begin{aligned} \bar{\alpha}(f) &= \sup\left\{\int_a^b f\, d\gamma : \gamma \in \sum(\gamma_0)\right\} = \inf\left\{\int_a^b P\, d\gamma_0 : P \in M_n, P \geqslant f\right\} \\ \underline{\alpha}(f) &= \inf\left\{\int_a^b f\, d\gamma : \gamma \in \sum(\gamma_0)\right\} = \sup\left\{\int_a^b P\, d\gamma_0 : P \in M_n, P \leqslant f\right\} \end{aligned} \tag{1.1}$$

where $M_n = sp(h_1, \ldots, h_n)$.

This duality principle was discovered independently by Isii [*2*] and Karlin ([*3*], p. 472).

A duality theorem very similar to the above is the following: Let X be a normed linear space with a partial ordering $\leqslant$ and K the positive cone of X. If Y is a subspace of X and L_0 a positive functional in X^* we denote by $\mathscr{L}(L_0)$ the class of all positive functionals in X^* which agree with L_0 on Y. If $x_0 \in X$ and $(\mathrm{int}(K)) \cap Y \neq \phi$, we have

$$\begin{aligned} \bar{\beta}(x_0) &= \sup\{L(x_0) : L \in \mathscr{L}(L_0)\} = \inf\{L_0(y) : y \in Y, y \geqslant x_0\} \\ \underline{\beta}(x_0) &= \inf\{L(x_0) : L \in \mathscr{L}(L_0)\} = \sup\{L_0(y) : y \in Y, y \leqslant x_0\} \end{aligned} \tag{1.2}$$

Also, there exist functionals $\bar{L}$, $\underline{L} \in \mathscr{L}(L_0)$ for which $\bar{L}(x_0) = \bar{\beta}(x_0)$, $\underline{L}(x_0) = \underline{\beta}(x_0)$. The proof of (1.2) follows the general line of proofs of duality theorems (see [1]). The condition $(\mathrm{Int}(K)) \cap Y \neq \phi$ guarantees that every positive functional on Y can be extended to a positive functional on X.

Taking X to be the space of all regularized functions on $[a,b]$ (i.e., $f \in X$ if and only if f is the uniform limit of step functions) with the natural ordering, and the supremum norm, we have that X^* is the space of all Borel measures on $[a,b]$. Thus if f is a regularized function, (1.2) recovers (1.1). It is interesting to note that in this case (1.2) also guarantees the existence of positive Borel measures $\bar{\gamma}, \underline{\gamma} \in \sum (\gamma_0)$, for which

$$\int_a^b f\,d\bar{\gamma} = \bar{\alpha}(f),\ \int_a^b f\,d\underline{\gamma} = \alpha(f).$$

It does not appear that the general case of (1.1) can be obtained from (1.2). However, by considering the space of all bounded Borel measurable functions with the supremum norm and the usual ordering, we have

$$\begin{aligned} \bar{\beta}(f) &= \sup\left\{\int_a^b f\,d\lambda : \lambda \in \Lambda(\lambda_0)\right\} = \inf\left\{\int_a^b P\,d\lambda_0 : P \in M_n,\ P \geqslant f\right\} \\ \underline{\beta}(f) &= \inf\left\{\int_a^b f\,d\lambda : \lambda \in \Lambda(\lambda_0)\right\} = \sup\left\{\int_a^b P\,d\lambda_0 : P \in M_n,\ P \leqslant f\right\} \end{aligned} \tag{1.3}$$

and there exists $\bar{\lambda}$, $\underline{\lambda} \in \Lambda(\lambda_0)$, such that $\int_a^b f\,d\bar{\lambda} = \bar{\beta}(f)$, $\int_a^b f\,d\lambda = \beta(f)$ where M_n is as before, λ_0 is a positive-bounded additive set function defined on the Borel sets and $\Lambda(\lambda_0)$ is the class of all positive bounded additive set functions defined on the Borel sets which agree with λ_0 on M_n. It is clear that (1.1) and (1.3) show that $\bar{\alpha}(f) = \bar{\beta}(f)$, $\underline{\alpha}(f) = \underline{\beta}(f)$, when λ_0 is countably additive.

The moment problem corresponding to (1.1) has been studied extensively and has many applications in analysis and statistics. An excellent account of these applications can be found in the last three chapters of Karlin and Studden [3]. The dual problem, which we shall call the problem of one-sided approximation, has not been systematically studied although some results on this problem have been obtained in connection with the moment problem ([3], p. 424). It is this problem of one-sided approximation which we will consider in this paper. It should be noted that because of the duality relations (1.1), it is possible to derive some of our results directly from the existing theory of moments.

Let μ be a positive Borel measure defined on $[a,b]$. If $h_1, \ldots, h_n$ is a Čebyšev system on $[a,b]$, $M_n = sp(h_1, \ldots, h_n)$ and f is a real-valued function defined on $[a,b]$, we denote by $M_n(f)$ the class of all these functions h in M_n which satisfy $h(x) \leqslant f(x)$ for all $x \in [a,b]$. We say that $h^* \in M_n(f)$ is a best one-sided approximation to f from below if

$$\int_a^b h^*\,d\mu = \sup\left\{\int_a^b h\,d\mu : h \in M_n(f) = \alpha(f)\right\}$$

Best one-sided approximations from above are defined similarly. All results will be stated for one-sided approximation from below although analogous results hold for one-sided approximation from above.

The case when $\mu(x) = x$ is of particular interest and has been studied by many authors. Error estimates for one-sided approximation by trigonometric polynomials have been obtained by Freud [*4*, *5*] and Ganelius [*6*]. Similar results for spline functions were obtained by Sharma and Meir [*7*]. A systematic study of one-sided approximation by algebraic polynomials was given by Bojanic and the author [*8*] and has served as a basis for the material presented here.

2. Preliminaries

The functions $h_1, \ldots, h_n$ are said to be a Čebyšev system on $[a,b]$ if they are continuous and linearly independent on $[a,b]$ and satisfy the following property:

PROPERTY C.1. *If $h \in M_n = sp(h_1, \ldots, h_n)$ has n distinct zeroes in $[a,b]$, then it is identically zero on $[a,b]$.*

In this case, M_n is called a Čebyšev space of degree n. We now state some well-known properties of Čebyšev systems which we shall need, the proofs of which can be found in [*9*].

PROPERTY C.2. *If $h_1, \ldots, h_n$ is a Čebyšev system and $x_1, \ldots, x_n$ are any distinct points of $[a,b]$, then the determinant*

$$D(x_1, \ldots, x_n) = \begin{vmatrix} h_1(x_1) \ldots h_n(x_1) \\ \vdots \\ h_1(x_n) \ldots h_n(x_n) \end{vmatrix}$$

is not zero.

PROPERTY C.3. *If $h_1, \ldots, h_n$ is a Čebyšev system on $[a,b]$, $x_1, \ldots, x_n$ distinct points of $[a,b]$ and $c_1, \ldots, c_n$ are arbitrary real numbers, then the system of equations*

$$a_1 h_1(x_k) + \ldots + a_n h_n(x_k) = c_k \qquad k = 1, \ldots, n$$

has a unique solution.

If a function f has a zero at the point $x_0 \in (a,b)$ and does not change sign at x_0, we shall say that f has a double zero at x_0. We denote by $Z(f)$ the number of zeroes of f on $[a,b]$, double zeroes being counted twice. The proof of the following properties of Čebyšev systems can be found in ([*3*], p. 30) and [*10*] respectively.

PROPERTY C.4. *Let $h_1, \ldots, h_n$ be a Čebyšev system on $[a,b]$. If $x_1, \ldots, x_m$ are m points in $[a,b]$ and $x_{j_1}, \ldots, x_{j_{m'}}$ is a subset of these points with $m + m' \leqslant n - 1$, then there is a function h in M_n which has zeroes at $x_1, \ldots, x_m$ and only at these points on (a,b) and changes sign at each $x_k \neq a, b, x_{j_1}, \ldots, x_{j_{m'}}$ and only at these points on $[a,b]$.*

PROPERTY C.5. *Let $h_1, \ldots, h_n$ be a Čebyšev system on $[a,b]$. If h is any function from M_n with $Z(h) \geqslant n$, then h is identically zero on $[a,b]$.*

The following interpolation property is a simple consequence of Property C.1.

PROPERTY C.6. *Let $h_1, \ldots, h_n$ be a Čebyšev system on $[a,b]$. If $x_1, \ldots, x_n$ are any n distinct points of $[a,b]$, we have the following interpolation formula with nodes at $x_1, \ldots, x_n$,*

$$h(x) = \sum_{k=1}^{n} h(x_k)\, l_k(x),$$

which holds for all functions h in M_n, where the function l_k is that function in M_n which has the value zero at each $x_j \neq x_k$ and the value one at the point x_k.

Suppose the functions $h_1, \ldots, h_n$ are all continuously differentiable on $[a,b]$. If the space M_n is a Čebyšev space of degree n and the space $M_n' = sp(h_1', \ldots, h_n')$ is a Čebyšev space of degree $n-1$, we say that M_n is a differentiable Čebyšev space of degree n. Also, we say that $h_1, \ldots, h_n$ is a differentiable Čebyšev system.

We shall need the following property of differentiable Čebyšev systems which is a simple application of Rolle's theorem.

PROPERTY D.1. *Let $h_1, \ldots, h_n$ be a differentiable Čebyšev system on $[a,b]$. Suppose $x_1, \ldots, x_m$ are any m distinct points of $[a,b]$ and $x_{j_1}, \ldots, x_{j_{m'}}$ is a subcollection of these points with $m + m' = n$. If h is any function in M_n which has zeroes at $x_1, \ldots, x_m$ and its derivative h' has zeroes at $x_{j_1}, \ldots, x_{j_{m'}}$, then h is identically zero on $[a,b]$.*

For discontinuous functions the ordinary concept of a zero does not give any information as to the closeness of the function to zero in a neighborhood of that zero as it does for continuous functions. It is thus advantageous to introduce the concept of an essential zero. The point $x_0 \in [a,b]$ is said to be an essential zero of the function f if for every neighborhood N of x_0 and every $\epsilon > 0$ there is a point $x \neq x_0$, $x \in N$ such that $|f(x)| < \epsilon$. The proof of the following property of essential zeroes is a simple compactness argument.

PROPERTY E.1. *If $K \subseteq [a,b]$ is a closed set in which f has no essential zeroes, then there exists a $c > 0$ such that $|f(x)| \geqslant c$ on K.*

3. Existence and Uniqueness of Best One-Sided Approximations

3.1. *Existence of Best One-Sided Approximations*

The existence of best one-sided approximations can be established with very little restrictions on f. We denote by $L(\mu)$ the class of μ-integrable functions. Our first theorem shows that the best one-sided approximation to f exists whenever f is in $L(\mu)$ and bounded from below. We note that when f is not bounded from below, $M_n(f)$ is empty.

THEOREM 3.1. *If f is any function in $L(\mu)$ which is bounded from below, then there exists a best one-sided approximation to f from M_n.*

Proof. Let (g_m) be a sequence of elements from $M_n(f)$ satisfying

$$\int_a^b g_m \, d\mu \to \alpha(f) \qquad (m \to \infty). \tag{*}$$

Then

$$\int_a^b |g_m| \, d\mu \leqslant \int_a^b |f - g_m| \, d\mu + \int_a^b |f| \, d\mu \leqslant A.$$

Since M_n is a finite-dimensional linear space, the closed sphere of radius A in M_n is compact. Thus, there is a subsequence (g_{m_k}) and a function g in $L(\mu)$ such that the sequence (g_{m_k}) converges to f in $L(\mu)$. Since we are in a finite-dimensional space, the sequence (g_{m_k}) also converges in the supremum norm and thus uniformly to g.

Since each $g_m \in M_n(f)$, we have $g_{m_k} \leqslant f$ on $[a,b]$ for $k = 1, 2, \ldots$ and so $g \leqslant f$ on $[a,b]$. Thus, $g \in M_n(f)$. From (*) it follows that

$$\int_a^b g d\mu = \lim_k \int_a^b g_{m_k} d\mu = \alpha(f)$$

and thus g is a best one-sided approximation to f from M_n.

We shall now give an existence theorem in which f need not be integrable.

THEOREM 3.2. *Suppose $x_1, \ldots, x_n$ are n distinct points of $[a,b]$ such that for any function h in M_n we have*

$$\int_a^b h d\mu = \sum_{k=1}^n A_k h(x_k)$$

with $A_k > 0$, $k = 1, \ldots, n$.

If f is any measurable function which is bounded from below on $[a,b]$ and which has a finite value c_k at x_k, $k = 1, 2, \ldots, n$, then there exists a best one-sided approximation to f from M_n.

Remark. The existence of quadrature formulae for M_n will be discussed in Section 4.

Proof. Let (g_m) be a sequence of functions from $M_n(f)$ such that

$$\int_a^b g_m\, d\mu \to \alpha(f) \qquad (m \to \infty).$$

Since

$$g_m(x_k) \leqslant c_k,\ k = 1, 2, \ldots, n,$$

we have

$$\int_a^b g_m\, d\mu = \sum_{k=1}^n A_k g_m(x_k) \leqslant \sum_{k=1}^n A_k c_k \qquad m = 1, \ldots.$$

Since

$$M_n(f) \neq \phi, \text{ we have } -\infty < \alpha(f) < \infty.$$

We now show that the sequences $(g_m(x_k))$ are bounded from below for $k = 1, \ldots, n$. For suppose there is a subsequence $(g_{m_j}(x_{k_0}))$ which converges to $-\infty$ as $j \to \infty$. Then

$$\int_a^b g_{m_j}\, d\mu = \sum_{k=1}^n A_k g_{m_j}(x_k) \to -\infty \qquad (j \to \infty),$$

which is impossible since $\alpha(f) > -\infty$. Thus, the sequences $(g_m(x_k))$ are bounded from below, $k = 1, \ldots, n$.

Using the interpolation formula given in Property C.6 with nodes at the points $x_1, \ldots, x_n$, it is easy to show that

$$\sup_{a \leqslant x \leqslant b} |g_m(x)| \leqslant n \left(\max_{1 \leqslant k \leqslant n} |g_m(x_k)| \right) \left(\max_{1 \leqslant k \leqslant n} \sup_{a \leqslant x \leqslant b} |l_k(x)| \right) \leqslant M$$

for $m = 1, \ldots$. Thus the functions g_m lie in a compact subset of M_n. Therefore, it is possible to select a subsequence from (g_m) which converges uniformly on $[a,b]$ to a function g in M_n. It is now easy to see that g is a best one-sided approximation to f from M_n.

3.2. *Uniqueness of Best One-Sided Approximations*

Although the existence of best one-sided approximations was established under very general conditions, criteria for uniqueness of best one-sided approximations are much more restrictive.

We denote by $C(\mu)$ the support of μ and by $|C(\mu)|$ the number of points in $C(\mu)$, each interior point of $[a,b]$ being counted twice. We denote by $Z(f)$ the number of essential zeroes of f contained in $[a,b]$, each interior zero being counted twice.

LEMMA 3.1. *If $h^* \in M_n(f)$ is a best one-sided approximation to f from M_n, then $Z(f - h^*) \geqslant \min(|C(\mu)|, n)$.*

Proof. Suppose to the contrary that $Z(f - h^*) < \min(|C(\mu)|, n)$. Let $x_1, \ldots, x_{k_0}$ be all the essential zeroes of $f - h^*$ in $[a,b]$. Consider the nonnegative function h in M_n which has zeroes at the points $x_1, \ldots, x_{k_0}$ and only at these points in $[a,b]$. The existence of such a function is guaranteed by Property C.4. Since $|C(\mu)| > Z(f-h^*)$, we clearly have $\int_a^b h d\mu > 0$.

Let $y_1, \ldots, y_{n-k_0}$ be any other $n-k_0$ points of $[a,b]$ chosen so that $x_1, \ldots, x_{k_0}$, $y_1, \ldots, y_{n-k_0}$ form a set of n distinct points of $[a,b]$. If $c > 0$, denote by h_c that function in M_n which has the value c at each of the points $x_1, \ldots, x_{k_0}, y_1, \ldots, y_{n-k_0}$. It is easy to see from the interpolation formula given by Property C.6 that if $\epsilon > 0$, there is a $c_\epsilon > 0$ such that for $0 < c \leqslant c_\epsilon$

$$\sup_{a \leqslant x \leqslant b} |h_c(x)| \leqslant \epsilon.$$

Thus, there is a $c_0 > 0$ such that

$$\int_a^b h_{c_0} d\mu < \int_a^b h d\mu.$$

Now, the function $h - h_{c_0}$ is in M_n and

$$h(x_j) - h_{c_0}(x_j) = -c_0 < 0 \text{ for } j = 1, \ldots, k_0.$$

Thus, for each j there is a neighborhood $N(x_j)$ of x_j, on which $h - h_{c_0}$ is negative. Let

$$K = [a,b] \setminus \bigcup_{j=1}^{k_0} N(x_j).$$

Then, K is compact and $f - h^*$ has no essential zeroes in K and thus, by Property E.1, there is a positive constant η for which $f - h^* \geqslant \eta$ on K. Let $\delta > 0$ be so small that $\delta(h - h_{c_0}) \leqslant \eta$ on $[a,b]$. Since $f - h^* \geqslant 0$ on $[a,b]$ and $h - h_{c_0}$ is negative on each $N(x_j), j = 1, \ldots, k_0$, we have

$$\delta(h(x) - h_{c_0}(x)) \leqslant 0 \leqslant f(x) - h^*(x) \qquad x \in \bigcup_{j=1}^{k_0} N(x_j).$$

On the other hand

$$\delta(h(x) - h_{c_0}(x)) \leqslant \eta \leqslant f(x) - h^*(x) \qquad x \in K.$$

Hence

$$\delta(h(x) - h_{c_0}(x)) \leqslant f(x) - h^*(x) \qquad x \in [a,b]$$

and

$$\int_a^b \delta(h - h_{c_0}) d\mu > 0.$$

This contradicts the fact that h^* is a best one-sided approximation to f from M_n and the lemma is proved.

Using the above lemma, it is easy to prove the following uniqueness theorem:

THEOREM 3.3. *Let $h_1, \ldots, h_n$ be a differentiable Čebyšev system on $[a,b]$. If f is differentiable on (a,b) and $|C(\mu)| \geqslant n$ then the best one-sided approximation to f from M_n is unique*

Proof. We first observe that since the functions f and $h_1, \ldots, h_n$ are all continuous, the essential zeroes of any linear combination of these functions are ordinary zeroes.

Suppose that h_1^* and h_2^* are two best one-sided approximations to f from M_n. Then, it is clear that $\frac{1}{2}(h_1^* + h_2^*)$ is also a best one-sided approximation to f from M_n. Let x_0 be a zero of $f - \frac{1}{2}(h_1^* + h_2^*)$. Since $h_j^*(x_0) \leqslant f(x_0)$ for $j = 1, 2$, we have $h_1^*(x_0) = h_2^*(x_0) = f(x_0)$. Also, if x_0 is in the interior of $[a,b]$, then $f - h_j^*$ has a minimum at the point x_0, $j = 1, 2$. Thus, $f' - h_j^{*\prime}$ is zero at each zero which is in the interior of $[a,b]$, $j = 1, 2$. Therefore, $h_1^{*\prime}(x_0) = f'(x_0) = h_2^{*\prime}(x_0)$ for each zero x_0 which lies in the interior of $[a,b]$.

If m_1 is the number of zeroes of $f - \frac{1}{2}(h_1^* + h_2^*)$ in $[a,b]$ and m_2 is the number of zeroes of $f - \frac{1}{2}(h_1^* + h_2^*)$ in (a,b), we have by Lemma 3.1 that $Z(f - \frac{1}{2}(h_1^* + h_2^*)) = m_1 + m_2 \geqslant n$. Thus, $h_1^* - h_2^*$ has m_1 zeroes in $[a,b]$ and $h_1^{*\prime} - h_2^{*\prime}$ has m_2 zeroes in (a,b). By Property D.1 $h_1^* = h_2^*$, and the theorem is proved.

Combining the fact that the difference of the function and its best approximation has sufficiently many zeroes and the fact that the set of best approximations is convex, to obtain a uniqueness theorem is a standard argument in approximation theory and appears, for example, in the proofs of uniqueness theorems for both L_1 and uniform approximation. In both of these cases, uniqueness of best approximations from Čebyšev spaces of finite degree can be guaranteed for continuous functions. The essential difference is one-sided approximation is that the difference between f and its best one-sided approximation has only half as many zeroes, but the differentiability of f and the restriction that $h_1, \ldots, h_n$ be a differentiable Čebyšev system imply that these zeroes are double zeroes. Hence, again, this difference has sufficiently many zeroes, and the uniqueness theorem can be proved.

In the next chapter we shall develop quadrature formulae for Čebyšev systems which will allow us to examine the problem of uniqueness more closely. However, for the sake of completeness we shall now state a non-uniqueness theorem whose proof will be given in Section 4.

THEOREM 3.4. *Let $h_1, \ldots, h_n$ be a differentiable Čebyšev system and suppose there exists a differentiable function f for which $h_1, \ldots, h_n, f$ is a Čebyšev system. Then, there exists a continuous function which is differentiable at all but finitely many points of $[a,b]$, whose best one-sided approximation from M_n is not unique.*

4. QUADRATURE FORMULAE FOR ČEBYŠEV SYSTEMS

If there exist points $x_1, \ldots, x_m$ in $[a,b]$ and constants $A_1, \ldots, A_m$ such that for any function h in M_n we have

$$\int_a^b h\,d\mu = \sum_{k=1}^{m} A_k h(x_k),$$

we say that $x_1, \ldots, x_m$ are nodes of a quadrature formula for M_n. If we also have $A_k > 0$, ($A_k \geqslant 0$), $k = 1, \ldots, m$, then we say that $x_1, \ldots, x_m$ are nodes of a quadrature formula with positive (nonnegative) coefficients. A case of special interest is when there is an $m_1 \leqslant m$ such that for any quadrature formula with nodes $y_1, \ldots, y_l$ including $x_1, \ldots, x_{m_1}$, we have $l \geqslant m$ and $l = m$ if and only if the points set $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ are identical. In this case we say $x_1, \ldots, x_m$ are nodes of a quadrature formula of highest possible degree of precision with fixed nodes $x_1, \ldots, x_{m_1}$.

The theories of quadrature formulae and one-sided approximation are closely related. It is this relationship which we now examine.

Our first result shows that any n distinct points of $[a,b]$ are nodes of a quadrature formula for M_n.

LEMMA 4.1. *If $x_1, \ldots, x_n$ are any n distinct points of $[a,b]$, then there are n real numbers $A_1, \ldots, A_n$ such that for any h in M_n we have*

$$\int_a^b h d\mu = \sum_{k=1}^{n} A_k h(x_k).$$

Proof. The space M_n has dimension n and hence its dual $M_n{}^*$ also has dimension n. The functionals ρ_k given by the formula $\rho_k(h) = h(x_k)$ are linearly independent in M_n, by virtue of Property C.2. Therefore, any linear functional in $M_n{}^*$ can be written as a linear combination of the ρ_k's. Taking in particular the functional

$$L(h) = \int_a^b h d\mu,$$

we have the desired result.

Our next result shows one of the fundamental relationships between the theories of one-sided approximation and quadrature formulae.

THEOREM 4.1. *Let f be a measurable function and h^* a best one-sided approximation to f from M_n. If $f - h^*$ has precisely $m \leqslant n$ essential zeroes on $[a,b]$, then these essential zeroes are nodes of a quadrature formula for M_n with nonnegative coefficients.*

Proof. Let $x_1, \ldots, x_m$ be all the essential zeroes of $f - h^*$ in $[a,b]$ and let $x_{m+1}, \ldots, x_n$ be any other $n - m$ points of $[a,b]$ chosen so that $x_1, \ldots, x_n$ form a set of n distinct points of $[a,b]$. By Lemma 4.1, there are real numbers A_k, $k = 1, \ldots, n$, such that for any h in M_n we have

$$\int_a^b h d\mu = \sum_{k=1}^{n} A_k h(x_k).$$

We first show that $A_k = 0$ for $k = m + 1, \ldots, n$. Assume, to the contrary, that

$$\sum_{k=m+1}^{n} |A_k| > 0.$$

For $r > 0$, let g_r be the function in M_n satisfying

$$g_r(x_k) = -\frac{1}{r} \qquad k = 1, \ldots, m$$

and

$$g_r(x_k) = r \operatorname{sgn} A_k \qquad k = m+1, \ldots, n.$$

Then

$$\int_a^b g_r \, d\mu = -\frac{1}{r} \sum_{k=1}^{m} A_k + r \sum_{k=m+1}^{n} |A_k|$$

which is positive provided we pick r sufficiently large, say $r = r_0$.

The function g_{r_0} is in M_n and

$$g_{r_0}(x_k) = -\frac{1}{r_0} < 0 \leqslant f(x_k) - h^*(x_k) \qquad k = 1, \ldots, m.$$

Thus there is an open set N in the relative topology of $[a, b]$ for which we have $x_1, \ldots, x_m \in N$ and

$$g_{r_0}(x) < 0 \text{ for } x \in N.$$

Since K is compact and $f - h^*$ has no essential zeroes in K, by Property E.1 there is a $c > 0$ such that

$$f(x) - h^*(x) \geqslant c \quad \text{for } x \in K.$$

Let

$$\eta = c / \max_{a \leqslant x \leqslant b} |g_{r_0}(x)|.$$

Since $f^*(x) - h^*(x) \geqslant 0$ for all $x \in [a, b]$, we have

$$\eta g_{r_0}(x) \leqslant 0 \leqslant f(x) - h^*(x) \quad \text{if } x \in N$$

and

$$\eta g_{r_0}(x) \leqslant \eta \max_{a \leqslant t \leqslant b} |g_{r_0}(t)| = c \leqslant f(x) - h^*(x) \quad \text{if } x \in K.$$

Hence,

$$\eta g_{r_0}(x) \leqslant f(x) - h^*(x) \quad \text{for } x \in [a, b]$$

and

$$\int_a^b g_{r_0} \, d\mu > 0.$$

Thus, $h^* + g_{r_0}$ is a better approximation to f from M_n, which is impossible. Thus, $A_k{}^n = 0$, $k = m+1, \ldots, n$.

We now show $A_k \geqslant 0$, $k = 1, \ldots, m$. Suppose to the contrary, $A_{k_0} < 0$ for some k_0, $1 \leqslant k_0 \leqslant m$. For $r > 0$, let g_r be that function in M_n satisfying

$$g_r(x_k) = -1 \qquad k \neq k_0,$$

$$g_r(x_{k_0}) = -r;$$

then

$$\int_a^b g_r \, d\mu = \sum_{k=1}^{m} A_k g_r(x_k) = -\sum_{k \neq k_0} A_k + r|A_{k_0}|.$$

Thus if r is sufficiently large, say $r = r_0$, we have

$$\int_a^b g_{r_0} \, d\mu > 0.$$

Arguing as before, we find an $\eta > 0$ such that $\eta g_{r_0}(x) \leqslant f(x) - h^*(x)$, $x \in [a,b]$ which again affords a contradiction to h^* being a best one-sided approximation to f.

We now examine in what sense the quadrature formula given in Theorem 4.1 is of the highest possible degree of precision. Our next result is stated for a quadrature formula with positive coefficients. It is clear that any formula with nonnegative coefficients determines a formula with positive coefficients in the obvious manner.

LEMMA 4.2. *Let $x_1, \ldots, x_l$ be nodes of a quadrature formula for M_n with positive coefficients. If $l \geqslant m \geqslant 2l - n$, then the quadrature formula with nodes $x_1, \ldots, x_l$ is of the highest possible degree of precision with fixed nodes $x_1, \ldots, x_m$.*

Proof. Let $y_1, \ldots, y_{l'}$ be nodes of a quadrature formula for M_n and suppose the points $x_1, \ldots, x_m$ are among $y_1, \ldots, y_{l'}$. We must show $l' \geqslant l$ and $l' = l$ if and only if the point set $\{x_1, \ldots, x_l\}$ and $\{y_1, \ldots, y_{l'}\}$ are identical. For this purpose suppose $l' < l$ or $l' = l$ but the point sets are not identical. Since $2l - m \leqslant n$, there is a function h in M_n which is zero at each y_k, $k = 1, \ldots, l'$ and one at each x_k, $k = 1, \ldots, l$ which is not among $y_1, \ldots, y_{l'}$. Using the quadrature formula with nodes at $y_1, \ldots, y_{l'}$ we have $\int_a^b h \, d\mu = 0$, and using the quadrature formula with nodes at $x_1, \ldots, x_l$ we have $\int_a^b h \, d\mu > 0$, the desired contradiction.

We shall now illustrate how the results given thus far in this section can be used to prove the existence of quadrature formulae of highest possible degree of precision with certain fixed nodes. For our next theorem, we need to consider approximation from both below and above. If f is in $C[a,b]$ and $h_*(h^*)$ is a best one-sided approximation to f from below (above), we let $f_* = f - h_*$ $(f^* = f - h^*)$.

THEOREM 4.2. *Let $h_1, \ldots, h_n$ and $h_1, \ldots, h_n, f$ both be Čebyšev systems, and let $|C(\mu)| \geqslant n$. Then*

(i) *if n is even, there are points $a < x_1 < \ldots < x_{[(n+1)/2]} < b$ such that for any function h in M_n we have*

$$\int_a^b h \, d\mu = \sum_{k=1}^{[(n+1)/2]} A_k^n h(x_k)$$

where $A_k^n > 0$, $k = 1, \ldots, [(n+1)/2]$;

(ii) *if n is even, there are points* $a = x_1 < \ldots < x_{[(n+2)/2]} = b$ *such that for any function h in M_n we have*

$$\int_a^b h d\mu = \sum_{k=1}^{[(n+2)/2]} B_k^n h(x_k)$$

where $B_k^n > 0$, $k = 1, \ldots, [(n+2)/2]$;

(iii) *if n is odd, there are points* $a = x_1 < \ldots < x_{[(n+1)/2]} < b$ *such that for any function h in M_n we have*

$$\int_a^b h d\mu = \sum_{k=1}^{[(n+1)/2]} C_k^n h(x_k)$$

where $C_k^n > 0$, $k = 1, \ldots, [(n+1)/2]$;

(iv) *if n is odd, there are points* $a < x_1 < \ldots < x_{[(n+1)/2]} = b$ *such that for any function h in M_n we have*

$$\int_a^b h d\mu = \sum_{k=1}^{[(n+1)/2]} D_k^n h(x_k)$$

where $D_k^n > 0$, $k = 1, \ldots, [(n+1)/2]$.

In each case, the points $x_1, \ldots, x_m$ are the zeroes of either f^* or f_*.

Also, these quadrature formulae are of the highest possible degree of precision with no fixed nodes, a and b fixed, a fixed, and b fixed respectively.

Proof. We consider only the case when $n = 2l$, the case when n is odd is handled similarly. We first show the existence of a quadrature formula with l nodes all in (a,b) and of one with a, b and other $l-1$ points as nodes. By virtue of Theorem 4.1, the zeroes of $f_*(f^*)$ are nodes of a quadrature formulae with nonnegative coefficients.

From Lemma 3.1, it follows that $Z(f_*) \geqslant n(Z(f^*) \leqslant n)$. Thus $Z(f_*) = n$, $Z(f^*) = n$. The functions f_* and f^* cannot have a common zero, for if they did, by Lemma 4.2 their zeroes would be identical and thus $Z(h_* - h^*) = Z(f_* - f^*) = n$. But then $f_* = f^*$, and since $f_* \geqslant 0$ on $[a,b]$ and $f^* \leqslant 0$ on $[a,b]$, we would have $f_* = f^* = 0$, which contradicts the linear independence of $h_1, \ldots, h_n, f$. Thus, the functions f_* and f^* have no common zero. Therefore, one of the functions f_*, f^* has exactly l zeroes, all of which are in (a,b), while the other function has a, b and $l-1$ points of (a,b) as zeroes. This proves the existence of the quadrature formulae.

The positivity of the coefficients of these formulae follows immediately from the fact that the kth coefficient is equal to $\int_a^b g_k d\mu$ where g_k is a non-negative function in M_n which has the value one at x_k and zero at $x_j \neq x_k$. We have $\int_a^b g_k d\mu > 0$ since $|C(\mu)| \geqslant n$. Finally, these formulae are of the highest possible degree of precision by virtue of Lemma 4.2.

The existence of the quadrature formulae given in Theorem 4.2 was first established by Krein [*10*]. Krein established these formulae for arbitrary Čebyšev systems. It is clear our method is only applicable to extendable Čebyšev systems. However, it is the nature of the nodes of these formulae as zeroes of f_* or f^* which is significant in the theory of one-sided approximations, as will be shown in Section 5.

We conclude this section with the proof of Theorem 3.4 which was stated without proof in Section 3.

Proof of Theorem 3.4. Let g be any function in M_n which is not identically zero, changes sign at each of the zeroes of the function $(h_{n+1})_*$ which is in the interior of $[a,b]$, and has a zero at each of the end points which is a zero of $(h_{n+1})_*$. Consider the function

$$f(x) = \begin{cases} g(x) & \text{when } g(x) \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $g(x)$ is differentiable and has only a finite number of sign changes, f is continuous on $[a,b]$ and differentiable except for a finite number of points in $[a,b]$.

We now show that there is more than one best one-sided approximation to f from M_n. First, by the quadrature formula of Theorem 4.2 which is based on the zeroes $x_1, \ldots, x_m$ of $(h_{n+1})_*$, we have for any function in $M_n(f)$

$$\int_a^b h d\mu = \sum_{k=1}^m A_k{}^n h(x_k) \leqslant \sum_{k=1}^m A_k{}^n f(x_k) = 0.$$

However, for any $0 < c < 1$, we have $cg(x) \leqslant f(x)$ for all $x \in [a,b]$ and

$$\int_a^b cgd\mu = \sum_{k=1}^m A_k{}^n cg(x_k) = 0.$$

Therefore, cg is a best one-sided approximation to f from M_n whenever $0 \leqslant c \leqslant 1$, and the theorem is proved.

5. Determination of Best One-Sided Approximations

In Section 4 we have shown that if h^* is a best one-sided approximation to f from M_n such that $f - h^*$ has n essential zeroes in $[a,b]$, then these zeroes are nodes of a quadrature formula for M_n with nonnegative coefficients. The converse of this statement is also true.

THEOREM 5.1. *Let $h^* \in M_n(f)$ and let $x_1, \ldots, x_m$ be essential zeroes of $f - h^*$. If $x_1, \ldots, x_m$ are nodes of a quadrature formula for M_n with nonnegative coefficients, then h^* is a best one-sided approximation to f from M_n.*

Proof. Let $A_1, \ldots, A_m$ be the coefficients of the quadrature formula with nodes at $x_1, \ldots, x_m$. If $h \in M_n(f)$, it is easy to see that we must have $h(x_k) \leqslant h^*(x_k)$ for $k = 1, \ldots, m$. Thus

$$\int_a^b h d\mu = \sum_{k=1}^m A_k h(x_k) \leqslant \sum_{k=1}^m A_k h^*(x_k) = \int_a^b h^* \, d\mu,$$

and the theorem is proved.

For a particular function f, it may be possible to ascertain that certain points $y_1, \ldots, y_m$ must be zeroes of $f - h^*$ where h^* is a best one-sided approximation to f from M_n. If it is also possible to show that $f - h^*$ can have at most n other zeroes counting multiplicities, then by virtue of Lemma 4.2 the zeroes of $f - h^*$ are the nodes of the quadrature formula of highest possible degree of precision with fixed nodes $y_1, \ldots, y_m$. Thus h^* can be determined if this quadrature formula is known. We give the simplest example of this method.

THEOREM 5.2. *Let $|C(\mu)| \geqslant n$ and suppose $h_1, \ldots, h_n, f$ is a Čebyšev system on $[a,b]$. Let $x_1, \ldots, x_m$ be nodes of one of the formulae given in* Theorem 4.2. *Then there is a function h^* in M_n for which $f - h^*$ has a constant sign on $[a,b]$ and zeroes at $x_1, \ldots, x_m$. The function h^* is a best one-sided approximation to f from above or below according to whether the sign of $f - h^*$ is positive or negative.*

Remark 1. In the case that $h_1, \ldots, h_n$ is a differentiable Čebyšev system on $[a,b]$ and f is differentiable on (a,b), h^* is the uniquely-defined function in M_n which satisfies

$$h^*(x_k) = f(x_k) \qquad k = 1, \ldots, m$$

$$h^{*\prime}(x_k) = f'(x_k) \quad \text{for } x_k \in (a,b),\, k = 1, \ldots, m.$$

Remark 2. If h^* is a best one-sided approximation from above (below), then using the nodes of the other formula for Theorem 5.2 determines a best one-sided approximation to f from below (above).

Proof. Let $\underline{h}^*$ and $\bar{h}^*$ denote best one-sided approximations to f from M_n from below and above respectively. Using Theorem 4.2, we see that the zeroes of $f - \underline{h}^*$ are nodes of one of the formulae given in Theorem 4.2. Also, the zeroes of $f - \bar{h}^*$ are nodes of the other formula in Theorem 4.2. Taking h^* appropriately as either $\underline{h}^*$ or $\bar{h}^*$ we have proved the theorem and Remark 2. Remark 1 follows immediately from Theorem 3.3.

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